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On Critic Centres.

By Frank Morley.

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§1. The twelve lines each containing three of the nine inflexions of a cubic will intersect in twelve other points. For, taking any such line, through an inflexion on it three other lines can be drawn. We have thus ten of the lines; the other two contain the other six inflexions, and intersect in a point which is not an inflexion. From the twelve lines we get twelve points, which are clearly the critic centres or possible double points of all cubics having the same inflexions as the one considered.

For the canonical form

$$U \equiv x^3 + y^3 + z^3 + 6mxyz = 0,$$

the inflexions (I) are

$$\begin{array}{ccc} 0\ 1\ -1, & 0\ 1\ -\omega, & 0\ 1\ -\omega^2, \\ -1\ 0\ 1, & -\omega\ 0\ 1, & -\omega^2\ 0\ 1, \\ 1\ -1\ 0, & 1\ -\omega\ 0, & 1\ -\omega^2\ 0, \end{array}$$

where $1, \omega, \omega^2$ are cube roots of 1.

The twelve lines (L) are seen to be

$$\begin{array}{llll} x, & x + y + z, & \omega x + y + z, & \omega^2 x + y + z, \\ y, & x + \omega y + \omega^2 z, & x + \omega y + z, & x + \omega^2 y + z, \\ z, & x + \omega^2 y + \omega z, & x + y + \omega z, & x + y + \omega^2 z, \end{array}$$

each column of three lines containing the nine points I.

Now the harmonic polar of a point of inflexion, say $0, 1, -1$, is $y - z$. For the polar conic $y^2 + 2mzx = z^2 + 2mxy$ breaks up into this line and the inflexional tangent $y + z - 2mx$. Thus we have corresponding to I the nine harmonic polars

$$\begin{array}{ccc} y - z, & y - \omega^2 z, & y - \omega z, \\ -x + z, & -\omega^2 x + z, & -\omega x + z, \\ x - y, & x - \omega^2 y, & x - \omega y, \end{array}$$

so that the line corresponding to any point is got by writing the coordinates as coefficients, interchanging ω and ω^2. The points corresponding to the lines L (*i. e.* the critic centres) are seen to be

$$\begin{array}{llll} 1\,0\,0, & 1\,1\,1, & \omega^2\,1\,1, & \omega\,1\,1, \\ 0\,1\,0, & 1\,\omega^2\,\omega, & 1\,\omega^2\,1, & 1\,\omega\,1, \\ 0\,0\,1, & 1\,\omega\,\omega^2, & 1\,1\,\omega^2, & 1\,1\,\omega, \end{array}$$

so that the point corresponding to any line is got by writing the coefficients as coordinates, interchanging ω and ω^2. From the symmetry the critic centres might be called the harmonic poles of the corresponding lines of inflexions. It is clear from the tables that the three harmonic polars of collinear inflexions meet in a point, the harmonic pole of the line of inflexions (an obvious property which I have not seen stated); that reciprocally the four harmonic poles of concurrent lines of inflexions lie on a line, the harmonic polar of the point (just as four lines L pass through each point I). Each column of three points forms the same triangle as the corresponding column of the lines L; on the sides are the nine inflexions, and through the angles pass the nine harmonic polars.

The points which have x as polar are given by $y^2 + 2mzx = 0$, $z^2 + 2mxy = 0$; the chords of intersection are

$$y^2 + 2mzx - (z^2 + 2mxy),\ y^2 + 2mzx - \omega(z^2 + 2mxy),\ y^2 + 2mzx - \omega^2(z^2 + 2mxy) = 0;$$

hence the harmonic pole is one, the others are the triangle formed by the inflexional tangents and lie on the harmonic polars. The table of inflexional tangents is

$$\begin{array}{lll} -2mx + y + z, & -2mx + \omega^2 y + \omega z, & -2mx + \omega y + \omega^2 z, \\ x - 2my + z, & \omega x - 2my + \omega^2 z, & \omega^2 x - 2my + \omega z, \\ x + y - 2mz, & \omega^2 x + \omega y - 2mz, & \omega x + \omega^2 y - 2mz, \end{array}$$

and the polar of 1, 0, 0 with regard to the triangle T formed by the first row is $x = 0$, since $\frac{dT}{dy} = 0$ and $\frac{dT}{dz} = 0$; so that the harmonic pole is the pole of the line L with regard to the three inflexional tangents. There are two lines L through a harmonic pole, and two such points on a line L; the polar conic of such a point (*e. g.* $x^2 + 2myz$) will touch the lines through it at the other poles on the lines.

To show the effect on a curve of passing through critic centres, we may say (what is easily verified) that a quartic through the harmonic poles inflects at each, the three tangents at points in a column meeting on the quartic.

§2. The polar conic of a point $x_1y_1z_1$ or O is

$$x_1x^2+y_1y^2+z_1z^2+2m\,(x_1yz+y_1zx+z_1xy)=0\,.$$

Eliminating m between this and U, we get for the locus of contacts of tangents from O,

$$Q\equiv 3xyz\,(x_1x^2+y_1y^2+z_1z^2)=(x^3+y^3+z^3)(x_1yz+y_1zx+z_1xy)\,.$$

This quintic Q is seen to pass through O, the twelve harmonic poles, and the nine points I. Now the latter points are also inflexions on Q. For since

$$\frac{dQ}{dx}=3yz\,(x_1x^2+\ldots)+6x_1x^2yz-3x^2\,(x_1yz+\ldots)-(x^3+\ldots)(y_1z+z_1y)\,,$$

the tangent at $0,\ 1,\ -1$ is

$$x\,(y_1+z_1)=x_1\,(y+z)\,,$$

which passes through O. Eliminating x between this and Q,

$$\begin{aligned}&3\,(y_1+z_1)\,x_1yz\,(y+z)\{x_1^3\,(y+z)^2+(y_1+z_1)^2(y_1y^2+z_1z^2)\}\\&=\{x_1^3\,(y+z)^3+(y_1+z_1)^3(y^3+z^3)\}\{x_1\,(y_1+z_1)\,yz+x_1\,(y+z)(y_1z+z_1y)\},\end{aligned}$$

or dividing by $x_1\,(y+z)$,

$$\begin{aligned}&3x_1^3\,(y_1+z_1)\,yz\,(y+z)^2+3yz\,(y_1+z_1)^3(y_1y^2+z_1z^2)\\&-x_1^3\,(y_1+z_1)\,yz\,(y+z)^2-yz\,(y_1+z_1)^4(y^2-yz+z^2)\\&-x_1^3\,(y_1z+z_1y)(y+z)^3-(y_1+z_1)^3(y^3+z^3)(y_1z+z_1y)=0\,;\end{aligned}$$

the terms of which $y+z$ is not an apparent factor may be written

$$3yz\,(y_1+z_1)^3\{y_1y^2+z_1z^2+yz\,(y_1+z_1)\}-yz\,(y_1+z_1)^4(y^2+z^2+2yz)\,,$$

or $$3yz\,(y_1+z_1)^3(y+z)(y_1y+z_1z)-yz\,(y_1+z_1)^4\,(y+z)^2.$$

Dividing by $y+z$,

$$\begin{aligned}2x_1^3\,(y_1+z_1)\,yz\,(y+z)-x_1^3\,(y_1z+z_1y)(y+z)^2-(y_1+z_1)^3(y^2-yz+z^2)(y_1z+z_1y)&\\+\,3yz\,(y_1+z_1)^3(y_1y+z_1z)-yz\,(y_1+z_1)^4(y+z)&=0\,.\end{aligned}$$

The terms of which $y+z$ is not an apparent factor may be written

$$(y_1+z_1)^3\{3yz\,(y_1y+z_1z+y_1z+z_1y)-(y^2+z^2+2yz)(y_1z+z_1y)\}\,;$$

or $$(y_1+z_1)^3\{3yz\,(y+z)(y_1+z_1)-(y+z)^2(y_1z+z_1y)\}.$$

Dividing by $y+z$,

$$\begin{aligned}x_1^3\{2yz\,(y_1+z_1)-(y+z)(y_1z+z_1y)\}+(y_1+z_1)^3\{3yz\,(y_1+z_1)&\\-(y+z)(y_1z+z_1y)\}-yz\,(y_1+z_1)^4&=0\,,\end{aligned}$$

or $$2yz\,(y_1+z_1)-(y+z)(y_1z+z_1y)=0\,,$$

or $$(y-z)(y_1z-z_1y)=0\,.$$

Thus the lines joining O to I are inflexional tangents to Q, and meet Q again on the corresponding harmonic polars.

From the equation of Q it passes through the intersections of

$$x_1x^2 + y_1y^2 + z_1z^2 = 0,\quad x_1yz + y_1zx + z_1xy = 0,$$

which are the four fixed points on the polar conic of O with regard to U. If we write X, Y for the special cubics $x^3 + y^3 + z^3$, $3xyz$; X', Y' for the first polars of O with regard to them; X'', Y'' for the second polars; X''', Y''' for their values at O, then the quintic and its polars may be seen, by successive operations with $x_1\frac{d}{dx} + y_1\frac{d}{dy} + z_1\frac{d}{dz}$, to be

$$\begin{aligned} YX' &= XY', \\ YX'' &= XY'', \\ Y'X'' + YX''' &= X'Y'' + XY''', \end{aligned}$$

or, since there is the identity, as may be seen by ordinary algebra,

$$Y'X'' - YX''' \equiv X'Y'' - XY''',$$

the cubic is

$$YX''' = XY''' \text{ or } Y'X'' = X'Y'';$$

differentiating either, the conic and line are

$$\begin{aligned} Y'X''' &= X'Y''', \\ Y''X''' &= X''Y'''. \end{aligned}$$

From the first form for the cubic it goes through I and O, and from the second it meets the polar conic at the points $X' = 0$, $Y' = 0$, and the polar line at the point $X'' = 0$, $Y'' = 0$, which is the fixed point on the polar line of O with regard to U. The polar quartic passes through the same point (P suppose), so that OP is a double tangent to the quartic.

We have now to show that Q is the general form of a quintic through the 21 points. A quintic about the triangle of reference is

$$\begin{aligned} x^4(b_1y + c_1z) + y^4(c_2z + a_2x) + z^4(a_3x + b_3y) + x^3(e_1y^2 + f_1z^2) + y^3(f_2z^2 + d_2x^2) \\ + z^3(d_3x^2 + e_3y^2) + xyz(ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy) = 0. \end{aligned}$$

Putting $x = 0$,

$$yz(c_2y^3 + b_3z^3 + f_2y^2z + e_3yz^2)$$

must

$$\equiv yz(y^3 + z^3),$$

$$\therefore\ d_2,\ d_3,\ e_3,\ e_1,\ f_1,\ f_2 = 0,$$

and

$$c_2 = b_3 = \alpha,\ a_3 = c_1 = \beta,\ b_1 = a_2 = \gamma,$$

so that the curve is

$$\left.\begin{aligned} \alpha yz(y^3 + z^3) + \beta zx(z^3 + x^3) + \gamma xy(x^3 + y^3) \\ + xyz(ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy) \end{aligned}\right\} = 0$$

Substituting $1\,1\,1$, $\omega\,1\,1$, $\omega^2\,1\,1$,

$$\begin{aligned} 2(\alpha+\beta+\gamma)+a+b+c+2(f+g+h) &= 0, \\ 2(\alpha+\beta\omega+\gamma\omega)+a+b\omega+c\omega+2(f\omega+g\omega^2+h\omega^2) &= 0, \\ 2(\alpha+\beta\omega^2+\gamma\omega^2)+a+b\omega^2+c\omega^2+2(f\omega^2+g\omega+h\omega) &= 0, \end{aligned}$$

$$\therefore\; 2\alpha+a=0,$$

or

$$a=-2\alpha,\; b=-2\beta,\; c=-2\gamma,$$

$$\begin{aligned} \therefore\; f\omega^2+g+h &= 0, \\ f\omega+g+h &= 0, \\ f+g+h &= 0, \end{aligned}$$

$$\therefore\; g+h=0,\; h+f=0,\; f+g=0, \text{ or } f,\,g,\,h=0.$$

Hence the general form is

$$\alpha yz(y^3+z^3-2x^3)+\beta zx(z^3+x^3-2y^3)+\gamma xy(x^3+y^3-2z^3)=0,$$

which is clearly equivalent to Q.

We have not used the points $1\,\omega\,\omega^2$, $1\,\omega^2\,\omega$, and need not have used $1\,1\,1$; so that a quintic through I and nine of the harmonic poles goes through the other three. Moreover, it goes through them without being further defined, and still contains two parameters. There is here a remarkable anomaly, for, while twenty points determine a quintic, we have a quintic through twenty-one points with two disposable constants.

For the other intersections of two such quintics we have

$$\frac{x_1x^2+y_1y^2+z_1z^2}{x_1yz+y_1zx+z_1xy}=\frac{x^3+y^3+z^3}{3xyz}=\alpha \text{ say},$$

or

$$x_1(x^2-\alpha yz)+\ldots\ldots=0$$

and

$$x_2(x^2-\alpha yz)+\ldots\ldots=0,$$

$$\therefore\; \frac{x^2-\alpha yz}{y_1z_2-z_1y_2}=\ldots\ldots,$$

or since $x^3+y^3+z^3-3\alpha xyz=0$,

$$\begin{vmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}=0;$$

that is to say, the remaining four intersections of two quintics whose parametric points are O, O' lie on the line OO'. These will be fixed for quintics $Q+\lambda Q'$, for their parametric points will be on OO', but not for quintics whose parametric points are off OO'. Hence though quintics through nineteen fixed points pass in general through six other fixed points, in this case we have quintics through twenty-one points whose remaining intersections may lie on any line not through the points. (If the parametric point is one of the twenty-one points, it is easily seen that the quintic splits up.) The proof that the other points of intersection are fixed is evaded by the existence of the two parameters (see Salmon's Plane Curves, p. 18). As was seen above, the twelve harmonic poles count only as nine conditions in fixing the quintic.

§3. It is clear that most of §2 is true in general. Let U, V be two n-ics, and let ξ denote $a\dfrac{d}{dx}+b\dfrac{d}{dy}+c\dfrac{d}{dz}$. Eliminating λ from

$$U+\lambda V=0,$$
$$\xi U+\lambda\xi V=0,$$

we get

$$Q\equiv V\xi U-U\xi V=0,$$

a $(2n-1)$-ic through the n^2 intersections, and also the $3(n-1)^2$ critic centres, for which

$$\frac{U_1}{V_1}=\frac{U_2}{V_2}=\frac{U_3}{V_3}=\frac{\xi U}{\xi V}=\frac{U}{V}.$$

The first polar of a, b, c or O is

$$\xi Q\equiv V\xi^2U-U\xi^2V,$$

which passes through $U=0$, $V=0$. Hence the tangents to Q at these points go through O. The points of contact of the other tangents from O lie on

$$\xi V.\xi^2U-\xi U.\xi^2V=0,$$

which touches Q at O, the common tangent being

$$\xi^{n-1}V.\xi^nU-=0,$$

ξ^nU being a constant.

To get the νth polar we may expand $\xi^\nu(V\xi U-U\xi V)$, remembering that Leibnitz's theorem applies to ξ, or use the more convenient form

$$\xi^{\nu+1}(VU)-2\xi^\nu(\xi V.U)-=0,$$

where now in each expansion we stop half way, *i. e.* keep only the terms in which the power of ξ acting on the first factor $<$ that acting on the second. The equivalence of the methods is clear if we write down the expansions. The polar cubic has a singular property; its equation becomes

$$n\xi^{n-2}V.\xi^{n-1}U+3(n-2)\xi^{n-3}V.\xi^nU-=0,$$

where, as above, the — precedes terms got by interchanging U and V. Now identically

$$\xi^{n-2}V.\xi^{n-1}U - \xi^{n-3}V.\xi^{n}U - = 0,$$

for if we operate with ξ, the left vanishes, and it also vanishes if $x, y, z = 0$. Hence the polar cubic is either

$$\xi^{n-2}V.\xi^{n-1}U - = 0,$$

or

$$\xi^{n-3}V.\xi^{n}U \quad - = 0,$$

and passes through the intersections of

$$\xi^{n-1}U,\ \xi^{n-1}V;\ \xi^{n-2}U,\ \xi^{n-2}V;\ \xi^{n-3}U,\ \xi^{n-3}V.$$

Thus the fixed point P_1 on the polar line of O with regard to $U + \lambda V$, the four fixed points P_2 on the polar conic, and the nine fixed points P_3 on the polar cubic lie on a cubic through O; OP_1 is a tangent at O to the cubic, and OP_2 are tangents at P_2.

Let Q' be the curve obtained from $a'b'c'$ or O'. Then for the intersections

$$U/V = \xi U/\xi V = \xi' U/\xi' V.$$

A possible set of points is

$$U = 0,\ \xi U = 0,\ \xi' U = 0,$$

or

$$\begin{aligned} xU_1 + yU_2 + zU_3 &= 0, \\ aU_1 + bU_2 + cU_3 &= 0, \\ a'U_1 + b'U_2 + c'U_3 &= 0, \end{aligned}$$

whence

$$\begin{vmatrix} x & y & z \\ a & b & c \\ a' & b' & c' \end{vmatrix} = 0.$$

The remaining $(2n-1)^2 - n^2 - 3(n-1)^2$ or $2(n-1)$ intersections $\therefore$ lie on OO'. These are fixed for $Q + \lambda Q'$, but not for curves whose parametric points lie off OO'. Since $(2n-1)(n+1)$ conditions fix a curve Q, and besides the $n^2 + 3(n-1)^2$ we require two more, it appears likely that all such curves through all but $n^2 + 3(n-1)^2 + 2 - (2n-1)(n+1)$, or $(n-2)(2n-3)$, of the critic centres pass through the rest. I have not noticed a similar difficulty for curves of even degree; but, at least for the case of the critic centres, the accepted theory appears to require limitation.

HAVERFORD COLLEGE, PA., October, 1887.

NOTE.—Professor Cayley has kindly cleared up the difficulty stated above; it comes under Bacharach's addition to Cayley's original theorem. See Cayley, "On the Intersection of Curves," Math. Annalen, Vol. XXX (1887), pp. 85–90; Bacharach, *ib.* XXVI (1885), pp. 275–299. On the point in question, Professor Cayley writes: "In fact, four points may be represented by $P_1 = 0$, $Q_4 = 0$; we have the quintics $A_4P_1 + B_1Q_4 = 0$, $A_4'P_1 + B_1'Q_4 = 0$, meeting in these four points and in twenty-one other points; the quintic $A_4B_1' - A_4'B_1 = 0$ passes through these twenty-one points but not through the four points—or generally we have

$$\alpha(A_4P_1 + B_1Q_4) + \alpha'(A_4'P_1 + B_1'Q_4) + \gamma(A_4B_1' - A_4'B_1) = 0,$$

a quintic through the twenty-one points but not through the four points." Thus, when we have shown that two $(2n - 1)$-ics through the n^2 intersections and $3(n-1)^2$ critic centres of two n-ics meet again in points on a right line, we do not expect other $(2n-1)$-ics through the $n^2 + 3(n-1)^2$ points to share the other $2(n-1)$ points.